Announcements

1) Picking up the pace...

2) HW 6 due Monday

3) HW 7 worth 50 points

Lemma: Let $f: X \to Y$ where X and Y are sets,

1) If $\{S_\alpha\}_{\alpha \in I} \subseteq Y$,

$$f^{-1}\left(\bigcup_{\alpha \in I} S_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(S_\alpha)$$

2) If $\{S_\alpha\}_{\alpha \in I} \subseteq Y$,

$$f^{-1}\left(\bigcap_{\alpha \in I} S_\alpha\right) = \bigcap_{\alpha \in I} f^{-1}(S_\alpha)$$

3) If $\{S_\alpha\}_{\alpha \in I} \subseteq X$

$$f\left(\bigcup_{\alpha \in I} S_\alpha\right) \subseteq \bigcup_{\alpha \in I} f(S_\alpha)$$

4) If X and Y are metric spaces then

$f: X \to Y$ is continuous

if and only if

$f^{-1}(C)$ is closed in X

for all C closed in Y.

<u>proof</u>: HW #7.

Theorem: Let X and Y be metric spaces and $f: X \to Y$ continuous.

Then

1) If $K \subseteq X$ is compact, then $f(K)$ is compact

2) If $S \subseteq X$ is connected, then $f(S)$ is connected.

proof: 1) Suppose $K \subseteq X$ is compact

Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover for $f(K)$.

Then $K \subseteq \bigcup_{\alpha \in I} U_\alpha$, so

$$K \subseteq f^{-1}(K) \subseteq f^{-1}\left(\bigcup_{\alpha \in I} U_\alpha\right)$$

$$= \bigcup_{\alpha \in I} f^{-1}(U_\alpha) \quad \text{by lemma}$$

Since f is continuous,

$f^{-1}(U_\alpha)$ is open for all $\alpha \in I$, We have

$$K \subseteq \bigcup_{\alpha \in I} f^{-1}(U_\alpha),$$

So $\{f^{-1}(U_\alpha)\}_{\alpha \in I}$ is an open cover of K.

Since K is compact, $\exists\ \alpha_1, \alpha_2, \ldots, \alpha_n$ with

$$K \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})$$

Apply f to both sides to get

$$f(K) \subseteq f\left(\bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})\right)$$

$$\subseteq \bigcup_{i=1}^{n} f\left(f^{-1}(U_{\alpha_i})\right)$$

by lemma

$$= \bigcup_{i=1}^{n} U_{\alpha_i}$$

Hence $f(K)$ is compact

2) Suppose $S \subseteq X$ is connected. By contradiction, suppose $f(S)$ is disconnected.

Then $\exists\, A, B \subseteq Y$,

$f(S) = A \cup B$ and

$\bar{A} \cap B = \bar{B} \cap A = \emptyset$.

Then $S \subseteq f^{-1}(f(S)) = f^{-1}(A \cup B)$

$= f^{-1}(A) \cup f^{-1}(B)$

by lemma.

Again by lemma,

$$f^{-1}(\bar{A} \cap B) = f^{-1}(\bar{A}) \cap f^{-1}(B)$$

and $f^{-1}(\bar{B} \cap A) = f^{-1}(\bar{B}) \cap f^{-1}(A)$.

However, it is not always the case that $f^{-1}(\overline{T}) = \overline{f^{-1}(T)}$

(HW #7?).

But since f is continuous,

$f^{-1}(\bar{A})$ is closed by lemma.

$$f^{-1}(\bar{A}) \supseteq f^{-1}(A)$$

Since $\overline{f^{-1}(A)}$ is the smallest closed set containing $f^{-1}(A)$, we get $f^{-1}(\bar{A}) \supseteq \overline{f^{-1}(A)}$.

Therefore,

$$
\begin{aligned}
\overline{f^{-1}(A)} \cap f^{-1}(B) &\subseteq f^{-1}(\bar{A}) \cap f^{-1}(B) \\
&= f^{-1}(\bar{A} \cap B) \\
&= f^{-1}(\emptyset) \\
&= \emptyset
\end{aligned}
$$

Interchanging the roles of A and B, we get

$$\overline{f^{-1}(B)} \cap f^{-1}(A) = \emptyset,$$

so that $\{f^{-1}(A), f^{-1}(B)\}$ is a separation for S.

But S is connected, so this is a contradiction.

Therefore, $f(S)$ is connected.

$\square$

Corollary: If X is a metric space and $f: X \to \mathbb{R}$ with the usual metric is continuous, then if $K \subseteq X$ is compact, f attains its supremum and infimum on $f(K)$.

Proof: If $K \subseteq X$ is compact and $f: X \to \mathbb{R}$ is continuous, then

By the previous theorem, $f(K)$ is compact. But then in $\mathbb{R}$, $f(K)$ must then be closed and bounded.

Bounded: $\inf\{f(K)\}$ and $\sup\{f(K)\}$ exist.

Closed. The inf and sup must be in $f(K)$. $\square$

Comment: Since the sup and inf in the previous corollary are attained, you may write "max" in place of "sup" and "min" in place of "inf".

Example 1: (Thomae)

$f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & x \notin \mathbb{Q} \\ 1/n, & \text{if } x \in \mathbb{Q}/\{0\}, \\ & x = \frac{m}{n} \text{ in lowest terms} \end{cases}$$

for example, $f\left(\frac{2}{3}\right) = \frac{1}{3}$

$$f\left(\frac{4}{8}\right) = \frac{1}{2}$$

f is discontinuous $\forall$ $x \in \mathbb{Q}$ since

$f(x) \neq 0 \quad \forall \; x \in \mathbb{Q}$

and for any such x, by density of the irrationals, $\exists$ sequence $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R} \setminus \mathbb{Q}$, $x_n \to x$. If f were continuous, we'd have

$0 = f(x_n) \to f(x) \neq 0$,

which is nonsense!

This shows f is discontinuous at all $x \in \mathbb{Q}$.

However, f is continuous for all irrational numbers!

Answer to question (discontinuities)

How bad can the set of discontinuities get?

A set $S \subseteq \mathbb{R}$ is called an F_σ set if S is a countable union of closed sets (S may not be closed)

Theorem: (set of discontinuities)

If $f:\mathbb{R}\to\mathbb{R}$ is any function and

$$D=\{x\in\mathbb{R} \mid f \text{ is discontinuous at } x\},$$

then D is an F_σ set.

Observation: (uses Baire Category Theorem)

The irrational numbers are not an F_σ set!

So you can never make a function that is continuous at all $x \in \mathbb{Q}$ and discontinuous at all $x \in \mathbb{R} \setminus \mathbb{Q}$!